$$\frac{du}{dx} + \frac{d^2u}{dx^2}.\frac{h}{2} + \ \ldots\ldots = 2ax + b + ah.$$

"Passing to the limit," that is making h to be 0,

$$\frac{du}{dx} = 2ax + b.$$

In this manner, the usual difficulty is entirely avoided. In indicating this modification, however, we have followed the popular notion that learners of the Calculus must first attend to some characteristic demonstration. But really, on reconsidering the question, what does it amount to? We have denoted the indeterminate coefficient of h by the symbol

$$\frac{du}{dx},$$

have gone through the usual course of demonstration, and – reproduced the original definition. "Passing to the limit" has amounted to nothing more.

The important inference hence arises, that a mere definition of the derivative

$$\frac{du}{dx},$$

as the coefficient of h, will suffice at the beginning, without that most unfortunate initiation of "passing to the limit." We can therefore commend to writers of Algebra the insertion of the formal notation and first simple rules of the Calculus, as an extension of the method of indeterminate coefficients, with still more satisfaction and confidence.

One further remark,—most writers have preferred the "infinitesimals" of Leibnitz to the "flowing quantities" of Newton. But at the first, leaving these to future applications, will it not be simpler to regard the quantities as primitive and derivative only, like the binomial coefficients or terms? As some ingenious writer has remarked, the increments need only be so small, that to suppose them smaller would not change the character of the results.

INTEGRATION OF POLYNOMIAL DIFFERENTIALS GENERAL FORMULAE OF REDUCTION.

BY PROF. DASCOM GREENE, TROY, N. Y.

Reduction Formulae, whereby the integration of a given differential is made to depend on that of another of a more simple form, play an important part in the Integral Calculus. They are usually obtained by the

application of the formula for integration by parts, which is itself nothing more than a formula of reduction. This process, however, applied to polynomial differentials of more than two terms, is so tedious that the formulae for higher polynomials are seldom given in the text books.

In this paper I have established by a method of remarkable simplicity, general reductions which include the reduction formulae for all polynomial differentials, and from which those answering to a given polynomial can be immediately obtained by making the proper substitutions.

Let X represent any polynomial, that is,

$$(a) \qquad X = ax^h + bx^n + cx^r + \&c.,$$

in which a, b, c, h, n, r, &c. are any constants, then

$$dX = (hax^{h-1} + nbx^{n-1} + rcx^{r-1} + \&c.)dx,$$

and we have

$$d(x^m X^p) = x^{m-1} X^{p-1}(mXdx + pxdX)$$

$$(b) \qquad = x^{m-1} X^{p-1}[m(ax^h + bx^n + cx^r + \&c.)dx + p(hax^h + nbx^n + rcx^r + \&c.)dx]$$

$$= x^{m-1} X^{p-1}[(m + ph)ax^h dx + (m + pn)bx^n dx + (m + pr)cx^r dx + \&c.]$$

or

$$x^m X^p = a(m + ph)\int x^{m+h-1} dx X^{p-1} + b(m + pn)\int x^{m+n-1} dx X^{p-1}$$

$$(1) \qquad + c(m + pr)\int x^{m+r-1} dx X^{p-1} + \&c.$$

Resuming now equation (b), and eliminating ax^h by means of (a), we have

$$d(x^m X^p) = x^{m-1} X^{p-1}[mXdx + ph(X - bx^n - cx^r - \&c.)dx + pnbx^n dx + \&c.]$$

$$= x^{m-1} X^{p-1}[(m + ph)Xdx + p(n - h)bx^n dx + p(r - h)cx^r dx + \&c.]$$

or

$$x^m X^p = (m + ph)\int x^{m-1} dx X^p + bp(n - h)\int x^{m+n-1} dx X^{p-1}$$

$$(2) \qquad + cp(r - h)\int x^{m+r-1} dx X^{p-1} + \&c.$$

In a similar way, eliminating bx^n, cx^r, &c., successively, we shall find

$$x^m X^p = (m + pn)\int x^{m-1} dx X^p + ap(h - n)\int x^{m+h-1} dx X^{p-1}$$

$$(3) \qquad + cp(r - n)\int x^{m+r-1} dx X^{p-1} + \&c.$$

$$x^m X^p = (m + pr)\int x^{m-1}dxX^p + ap(h-r)\int x^{m+h-1}\,dxX^{p-1}$$

$$(4) \qquad\qquad + bp(n-r)\int x^{m+n-1}dxX^{p-1} + \&c.$$

&c. &c.

Equating the second members of (1) and (2), we have the identical equation

$$a(m+ph)\int x^{m+h-1}dxX^{p-1} + b(m+ph)\int x^{m+n-1}dxX^{p-1}$$

$$+ c(m+ph)\int x^{m+r-1}dxX^{p-1} + \&c. = (m+ph)\int x^{m-1}dxX^p$$

$$\text{or} \int x^{m-1}dxX^p = a\int x^{m+h-1}dxX^{p-1} + b\int x^{m+n-1}dxX^{p-1}$$

$$(5) \qquad\qquad + c\int x^{m+r-1}dxX^{p-1} + \&c.$$

These elegant symmetrical formulae are likewise entirely general, being true for all values of m, p, h, n, r, &c., positive or negative, entire or fractional. Two or three examples will show the facility with which they may be used in deriving formulae of reduction.

I. If we limit the number of terms of the polynomial to *three*, and make $h = 0$, and $r = 2n$, then

$$X = a + bx^n + cx^{2n},$$

and (1), (2), (3) and (4) reduce to

$$x^m X^p = am\int x^{m-1}dxX^{p-1} + b(m+pn)\int x^{m+n-1}dxX^{p-1}$$

$$(6) \qquad\qquad + c(m+2pn)\int x^{m+2n-1}dxX^{p-1}$$

$$x^m X^p = m\int x^{m-1}dxX^p + bpn\int x^{m+n-1}dxX^{p-1}$$

$$(7) \qquad\qquad + 2cpn\int x^{m+2n-1}dxX^{p-1}$$

$$x^m X^p = (m+pn)\int x^{m-1}dxX^p - apn\int x^{m-1}dxX^{p-1}$$

$$(8) \qquad\qquad + cpn\int x^{m+2n-1}dxX^{p-1}$$

$$x^m Xp = (m+2pn)\int x^{m-1}dxX^p - 2apn\int x^{m-1}dxX^{p-1}$$

$$(9) \qquad\qquad - bpn\int x^{m+n-1}dxX^{p-1}$$

Each of the equations (6), (7), (8) and (9), expresses a relation between three integrals, and being solved for each in succession, will furnish three formulae for the reduction of the integral of the trinomial differential $x^m dx(a+bx^n+cx^{2n})^p$. It is unnecessary to develope them in detail here.

II. If $n = 1$, the equations just obtained reduce to

$$X = a + bx + cx^2,$$

$$(10)\quad x^m X^p = am\int x^{m-1}dxX^{p-1} + b(m+p)\int x^m dxX^{p-1} + c(m+2p)\int x^{m+1}dxX^{p-1}$$

$$(11)\quad x^m X^p = m\int x^{m-1}dxX^p + bp\int x^m dxX^{p-1} + 2cp\int x^{m+1}dxX^{p-1}$$

$$(12)\quad x^m X^p = (m+p)\int x^{m-1}dxX^p - ap\int x^{m-1}dxX^{p-1} + cp\int x^{m+1}dxX^{p-1}$$

$$(13)\quad x^m X^p = (m+2p)\int x^{m-1}dxX^p - 2ap\int x^{m-1}dxX^{p-1} - bp\int x^m dxX^{p-1}$$

Each of these equations will give three reduction formulae for

$$\int x^m dx(a + bx + cx^2)^p.$$

III. If $c = 0$, equations (6), (7) and (8) become

$$(14)\quad x^m X^p = am\int x^{m-1}dxX^{p-1} + b(m+pn)\int x^{m+n-1}dxX^{p-1}$$

$$(15)\quad x^m X^p = m\int x^{m-1}dxX^p + bpn\int x^{m+n-1}dxX^{p-1}$$

$$(16)\quad x^m X^p = (m+pn)\int x^{m-1}dxX^p - apn\int x^{m-1}dxX^{p-1}$$

in which $X = a + bx^n$.

Solving (14), (15) and (16) for each integral in succession, and writing the results in their simplest form, we have the following well-known formulae for the reduction of binomial differentials :

$$(17)\quad \int x^{m-1}dxX^p = \frac{x^m X^{p+1}}{am} - \frac{b(m+pn+n)}{am}\int x^{m+n-1}dxX^p$$

$$(18)\qquad \int x^{m-1}dx X^p = \frac{x^{m-n}X^{p+1}}{b(m+pn)} - \frac{a(m-n)}{b(m+pn)}\int x^{m-n-1}dx X^p$$

$$(19)\qquad \int x^{m-1}dx X^p = \frac{x^m X^p}{m} - \frac{bpn}{m}\int x^{m+n-1}dx X^{p-1}$$

$$(20)\qquad \int x^{m-1}dx X^p = \frac{x^{m-n}X^{p+1}}{bn(p+1)} - \frac{m-n}{bn(p+1)}\int x^{m-n-1}dx X^{p+1}$$

$$(21)\qquad \int x^{m-1}dx X^p = \frac{x^m X^p}{m+pn} + \frac{apn}{m+pn}\int x^{m-1}dx X^{p-1}$$

$$(22)\qquad \int x^{m-1}dx X^p = -\frac{x^m X^{p+1}}{an(p+1)} + \frac{m+pn+n}{an(p+1)}\int x^{m-1}dx X^{p+1}.$$

It should also be observed that equations (14), (15) and (16) may be obtained directly by following the general method by which (1) and (2) were derived, and this method will be found much more simple than that generally pursued in obtaining reduction formulae for binomials.*

Erratum. On page 138, line 6, for "general reductions" read general relations.

USEFUL FORMULAE IN THE CALCULUS OF FINITE DIFFERENCES.

BY G. W. HILL.

The finding of the values of the differential coefficients of a function of a single variable and of the single and double integrals with respect to the independent variable, from special values of the function computed at equidistant intervals, is an operation very frequent in Planetary Astronomy. The following seems a simpler exposition of the matter than has hitherto been given :

Let y be a function of x computed for the series of values of x, ... $a-h$, a, $a+h$, $a+2h$, ...; and let the differences and first and second summed values of y be denoted thus,

....	....	....	....	....	....	....	....
$a-h$	$\Delta^{-2}y_{-1}$	$\Delta^{-1}y_{-\frac{3}{2}}$	y_{-1}	$\Delta y_{-\frac{3}{2}}$	$\Delta^2 y_{-1}$	$\Delta^3 y_{-\frac{3}{2}}$	....
a	$\Delta^{-2}y_0$	$\Delta^{-1}y_{-\frac{1}{2}}$	y_0	$\Delta y_{-\frac{1}{2}}$	$\Delta^2 y_0$	$\Delta^3 y_{-\frac{1}{2}}$	....
$a+h$	$\Delta^{-2}y_1$	$\Delta^{-1}y_{\frac{1}{2}}$	y_1	$\Delta y_{\frac{1}{2}}$	$\Delta^2 y_1$	$\Delta^3 y_{\frac{1}{2}}$	....
$a+2h$	$\Delta^{-2}y_2$	$\Delta^{-1}y_{\frac{3}{2}}$	y_2	$\Delta y_{\frac{3}{2}}$	$\Delta^2 y_2$	$\Delta^3 y_{\frac{3}{2}}$	....
....	....	....	....	....	....	....	....